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THE ROLLING OF A RIGID WHEEL ALONG A DEFORMABLE RAIL†

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A model of a rigid wheel which rolls without slipping along a viscoelastic rail (beam) lying on a viscoelastic base is considered. Since point contact is assumed between the rail and the wheel, in the steady state (the wheel rolls in the vertical plane at a constant velocity) the problem under consideration is similar to that of the vibrations of a beam under the action of a moving load [1]. The problem of rolling has been treated in different formulations in many publications [2–9].

1. MECHANICAL MODEL OF THE SYSTEM

Let a rigid wheel and a rail be arranged in the plane $O_1X_1Y_1$. We shall assume that the rail is a beam which undergoes pure flexure and lies on a viscoelastic base. We shall specify the kinetic energy and potential energy functional and the functional of the dissipative forces in the form

$$T=\frac{m}{2}(x_1^{\cdot 2}+y_1^{\cdot 2})+\frac{J}{2}\theta^{\cdot 2}+\frac{1}{2}\int_{-b}^{b}\rho w^{\cdot 2}ds$$

$$\Pi=\frac{1}{2}\int_{-b}^{b}(k_1w''^2+k_2w^2)ds-F(t)x_1+P(t)y_1-M(t)\theta \tag{1.1}$$

$$D=\frac{1}{2}\int_{-b}^{b}\chi(d_1w^{\cdot\,''2}+d_2w^{\cdot 2})ds$$

where m and J are the mass and moment of inertia of the wheel about its axis, x_1, y_1, θ are the coordinates of the centre of the wheel at the point O and the angle of its rotation, $w(s, t)$ ($s \in [-b, b]$) are the displacements of the points of the neutral line of the rail along the axis O_1Y_1, ρ is the linear density of the rail, d_1 and d_2 are measures of the internal friction (a Kelvin–Voight material), χ is a small parameter, k_1 and k_2 are the flexural stiffness of the rail and the stiffness of the base, and F, $-P$ and M are the external forces and moment applied to the wheel (Fig. 1); a dot denotes differentiation with respect to time t and a prime denotes differentiation with respect to the variable s.

We shall solve the problem assuming that the curvature of the line of contact of the rail and the wheel (the line l_1) is smaller than the curvature of the wheel. Contact between the wheel and the rail is then made at the single point K, and the line l_1 and the outer circumference of the wheel of radius r have a common tangent at the point of contact. We shall further assume that contact between the wheel and the rail takes place without slipping. In the undeformed state, the neutral line of the rail l_0 coincides with the axis O_1X_1 and the line of contact l_1 is given by the equation $Y_1 = h$. In the strained state, according to the hypothesis of plane sections, the points of the neutral line are determined by the vector, $\mathbf{R}_0 = s\mathbf{e}_1 + w(s, t)\mathbf{e}_2$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors along the axes O_1X_1 and O_1Y_1, respectively, and points of the line of contact are determined by the vector $\mathbf{R}_1 = (s - h\sin\alpha)\mathbf{e}_1 + (w(s, t) + h\cos\alpha)\mathbf{e}_2$ (Fig. 1).

Let the system of coordinates Oxy be rigidly associated with the wheel and let ϑ be the angular coordinate of the points of its rim. The angle ϑ_0 corresponds to the point of contact K. The angle $\theta + \varphi_0$ is close to $3\pi/2$ and, consequently, the angle $\alpha = \theta + \varphi_0 - 3\pi/2$ is small (Fig. 1). The contact conditions at the point K have the form

$$s_0=x_1+(r+h)\sin\alpha,\quad w_0=y_1-(r+h)\cos\alpha\quad (w_0=w(s_0,t)) \tag{1.2}$$

Here s_0 is the coordinate of the point of the neutral line corresponding to the point K on the line of contact l_1. From the contact conditions (1.2) we obtain relations which connect the possible displacements. We shall henceforth assume that the angle α is small. These relations then take the form

$$\delta x_1+r_1\delta\theta=0,\quad \delta w_0=\delta y_1+r_1\alpha\delta\theta\quad (r_1=r+h) \tag{1.3}$$

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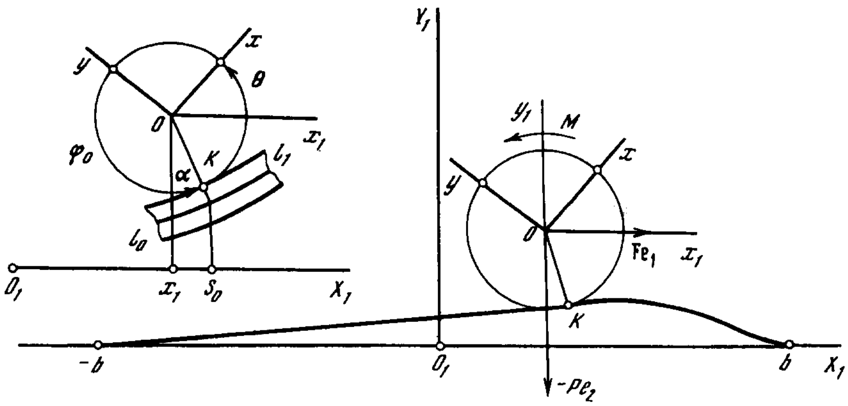

Fig. 1.

Conditions (1.2) actually mean that a wheel of radius r_1 rolls along the neutral line l_0. We shall also suppose that the reaction which occurs at the point of contact K is transferred to the corresponding point of the neutral line with coordinate s_0 without the addition of a moment proportional to the small quantity h/r.

2. THE EQUATIONS OF MOTION. THE STEADY-STATE CONDITION

We obtain the equations of motion of the system and the matching conditions at the point of contact from the Hamilton–Ostrogradskii variational principle

$$\delta\int_{t_1}^{t}(T-\Pi)dt-\int_{t_1}^{t_2}\int_{-b}^{b}\chi(d_1 w^{\cdot\prime\prime}\delta w''+d_2 w^{\cdot}\delta w)dsdt+$$

$$+\int_{t_1}^{t_2}[\mu(t)(\delta w_0-\delta y_1-r_1\alpha\delta\theta)+\nu(t)(\delta x_1+r_1\delta\theta)]dt=0 \tag{2.1}$$

where $\mu(t)$, $\nu(t)$ are Lagrangian multipliers. It follows from the condition for the existence of a potential energy functional in (1.1) that the function $w(s, t)$ and its first derivative with respect to s are continuous over the interval $[-b, b]$ and, in particular, at the point $s = s_0$. On splitting the range of variation of s into two parts: $[-b, s_0]$ and $[s_0, b]$ and integrating by parts, we find, from relations (2.1), the equations of motion and the matching conditions in the form

$$\begin{gathered}
m x_1^{\cdot\cdot}=F+\nu,\quad k_1[w'']_0+\chi d_1[w^{\cdot\prime\prime}]_0=0\\
m y_1^{\cdot\cdot}=-P-\mu,\quad k_1[w''']_0+\chi d_1[w^{\cdot\prime\prime\prime}]_0=\mu\\
J\theta^{\cdot\cdot}=M-\mu\xi_0+\nu r_1,\quad [w]_0=[w']_0=0
\end{gathered} \tag{2.2}$$

$$\rho w^{\cdot\cdot}+k_1 w^{\prime v}+\chi d_1 w^{\cdot\prime v}+k_2 w+\chi d_2 w^{\cdot}=0,\quad s\neq s_0$$

Here, $[f(s, t)]_0 = f(s_0 + 0, t) - f(s_0 - 0, t)$ is the discontinuity in the function when $s = s_0$, $\xi_0 = r_1\alpha$. Relations (2.2) form a complete system of equations from which the motion of the system can be determined when account is taken of the matching equations (1.2), the boundary conditions $w(\pm, t) = w'(\pm b, t) = 0$ and, also, the condition of coincidence of the tangents to the wheel and the line of contact l_1 at the point K and the condition for rolling without slipping.

We consider the rolling of the wheel without slipping at a constant velocity c when $x_1^{\cdot} = c, y_1 = \text{const}$, $\theta^{\cdot} = \text{const}$, $w(s, t) = W(\xi)$, $\xi = s - ct$, $\xi_0 = \text{const}$. We shall initially consider small dissipative forces and neglect them by putting $\chi = 0$. Moreover, by putting $b \gg |x_1|$ in the time interval being considered, we replace the boundary conditions at $s = \pm b$ by the condition $w(\pm\infty, t) = w'(\pm\infty, t) = 0$. In the steady state, the neutral line l_0 is independent of time in the moving system of coordinates Ox_1y_1, which is gradually displaced at a velocity c along the axis O_1X_1. In this case, we represent relations (2.2) in the form

$$F+\nu=0,\ P=-\mu,\ M+\nu r_1=\mu\xi_0,\ [W''']_0=\mu/k_1 \tag{2.3}$$

$$\rho c^2 W''+k_1 W^{\prime v}+k_2 W=0,\ [W]_0=[W']_0=[W'']_0=0$$

since

$$\frac{\partial w(s,t)}{\partial t}=-c\frac{\partial W(\xi)}{\partial \xi},\quad \frac{\partial^2 w(s,t)}{\partial t^2}=c^2\frac{\partial^2 W(\xi)}{\partial \xi^2},\quad \frac{\partial^n w(s,t)}{\partial s^n}=\frac{\partial^n W(\xi)}{\partial \xi^n}$$

The discontinuities in the function $W(\xi)$ and its derivatives in (2.3) are calculated when $\xi = \xi_0$.

The neutral line of the rail is determined in the form (D_n are the roots of the characteristic equation)

$$W(\xi)=\sum_{n=1}^{4} C_n \exp(D_n\xi),\quad D_n=\pm\left(\frac{-\rho c^2\pm\Delta}{2k_1}\right)^{1/2},\quad \Delta=(\rho^2c^4-4k_1k_2)^{1/2} \tag{2.4}$$

The roots D_n are equal to $\pm\varepsilon\pm i\omega$ if $0 \leqslant c < c^*$ and $\pm i\omega_1$, $\pm i\omega_2$ if $c > c^*$ where $c^* = (4k_1k_2/\rho^2)^{1/4}$.

The critical velocity c^* separates the two domains where the behaviour of the rail is qualitatively different when the wheel rolls. This has been noted in a study of the dynamics of a beam with moving loads [1, 10]. In the domain $c < c^*$, the functions

$$W(\xi)=\begin{cases} C_1\exp[(\varepsilon+i\omega)\xi]+\overline{C}_1\exp[(\varepsilon-i\omega)\xi], & \xi<\xi_0 \\ C_2\exp[(-\varepsilon-i\omega)\xi]+\overline{C}_2\exp[(-\varepsilon+i\omega)\xi], & \xi>\xi_0 \end{cases} \tag{2.5}$$

and $W(\pm\infty) = W'(\pm\infty) = 0$. Here, C_1 and C_2 are arbitrary constants and a bar over a symbol denotes at complex conjugate quantity. In the domain $c > c^*$, it is impossible to satisfy the conditions at infinity, and the problem has to be investigated taking account of dissipative forces ($\chi \neq 0$). The case of subcritical velocities is important in practice since the critical velocity is of the order of 1500 km /hour. We merely note that drag occurs when a wheel rolls at velocities exceeding the critical velocity.

When account is taken of (2.5), the matching conditions (2.3) have the form

$$\begin{aligned} &\sum_{k=1}^{4} D_k^n Z_k=-\frac{\mu}{k_1}\delta_{3n},\quad n=0,1,2,3 \\ &Z_1=\overline{Z}_2=C_1\exp(D_1\xi_0),\quad Z_3=\overline{Z}_4=-C_2\exp(D_3\xi_0) \\ &D_1=\overline{D}_2=\varepsilon+i\omega,\quad D_3=\overline{D}_4=-\varepsilon-i\omega \end{aligned} \tag{2.6}$$

From the system of linear equations (2.6) in Z_k, we obtain

$$Z_1+Z_2=\mu/\zeta,\quad \zeta=4k_1\varepsilon(\varepsilon^2+\omega^2),\quad D_1Z_1+D_2Z_2=0$$

The approximate equation of a circle of radius r_1 with its centre at the point O in the neighbourhood of its point of intersection with the negative part of the axis Oy_1 has the form

$$Y_1=y_1-r_1+\xi^2/(2r_1)$$

This circle touches the neutral line at $\xi = \xi_0$ and, consequently

$$y_1-r_1+\xi_0^2/(2r_1)=Z_1+Z_2=\mu/\zeta,\quad \xi_0/r_1=D_1Z_1+D_2Z_2=0$$

On taking account of relation (2.3), we find

$$\xi_0=0,\quad y_1-r_1=-P/\zeta,\quad M-r_1F=0$$

Steady-state motion with a velocity c is possible if $M = r_1F$, P = const. The neutral line of the rail is defined by the equation

$$W(\xi)=-\frac{P}{\zeta}\times\begin{cases} \exp(\varepsilon\xi)(\omega\cos\omega\xi-\varepsilon\sin\omega\xi), & \xi<0 \\ \exp(-\varepsilon\xi)(\omega\cos\omega\xi+\varepsilon\sin\omega\xi), & \xi>0 \end{cases} \tag{2.7}$$

In the case of motion at a velocity greater than c^*, the small dissipative forces are taken into account. The characteristic equation takes the form

$$c\chi d_1D^5-k_1D^4-\rho c^2D^2+c\chi d_2D-k_2=0$$

and, for small χ, has a root

$$D_5 = \frac{k_1}{c\chi d_1} + \frac{\rho c^3 d_1}{8k_1^2}\chi + o(\chi)$$

and the roots $\pm i\omega_1$, $\pm i\omega_2$ produce the corrections

$$\Delta_i = \frac{\chi c(d_1\omega_i^4 + d_2)}{2\rho c^2 - 4k_1\omega_i^2} + o(\chi), \quad i = 1,2$$

Further

$$\omega_i^2 = (\rho c^2 \mp \Delta)/(2k_1), \quad \omega_1 < \omega_2$$
$$\Delta_i = \pm\chi c(d_1\omega_i^4 + d_2)/(2\Delta), \quad i = 1,2; \quad \Delta_1 > 0, \quad \Delta_2 < 0$$

Consequently, the neutral line of the strained rail is represented in the form

$$W(\xi) = \begin{cases} C_1 \exp[(\Delta_1 + i\omega_1)\xi] + \overline{C}_1 \exp[(\Delta_1 - i\omega_1)\xi] + C_3 \exp(D_5\xi), & \xi < \xi_0 \\ C_2 \exp[(\Delta_2 + i\omega_2)\xi] + \overline{C}_2 \exp[(\Delta_2 - i\omega_2)\xi], & \xi > \xi_0 \end{cases} \tag{2.8}$$

The coefficients C_k in (2.8) are determined from the matching conditions

$$[W^{(n)}]_0 = 0, \quad n = 0,1,2,3; \quad [W^{(4)}]_0 = -\mu/(c\chi d_1) \tag{2.9}$$

The second derivative of the function $W(\xi)$ at the point of contact K is continuous since it follows from the condition for the existence of the dissipative functional (1.1) that the function $W(\xi)$ belongs to the Sobolev space $W_2^3([-b, b])$. From conditions (2.9), when (2.8) is taken into account, we obtain a system of five linear equations in the quantities U_k ($k = 1, \ldots, 5$)

$$\sum_{k=1}^{5} D_k^n U_k = \frac{\mu}{c\chi d_1}\delta_{4n}, \quad n = 0,\ldots,4$$
$$U_1 = \overline{U}_2 = C_1 \exp[(\Delta_1 + i\omega_1)\xi_0], \quad U_3 = \overline{U}_4 = -C_2 \exp[(\Delta_2 + i\omega_2)\xi_0]$$
$$U_5 = C_3 \exp(D_5\xi_0), \quad D_1 = \overline{D}_2 = \Delta_1 + i\omega_1, \quad D_3 = \overline{D}_4 = \Delta_2 + i\omega_2$$

Next, in much the same way as in the subcritical case, we find

$$W(\xi_0) = y_1 - r_1 + \frac{\xi_0^2}{2r_1} = -U_3 - U_4 = \frac{2\mu\chi c(d_1\omega_1^2\omega_2^2 + d_2)}{k_1^2(\omega_2^2 - \omega_1^2)^3} + o(\chi)$$
$$W'(\xi) = \frac{\xi_0}{r_1} = -D_3U_3 - D_4U_4 = -\frac{\mu}{k_1(\omega_2^2 - \omega_1^2)} + O(\chi)$$

Since $\mu = -P$ and $\omega_2^2 > \omega_1^2$, then $W(\xi_0) < 0$, but $W'(\xi_0) > 0$. According to (2.3)

$$M - r_1F = -P\xi_0 = -\frac{r_1P^2}{k_1(\omega_2^2 - \omega_1^2)} + o(\chi)$$

If we put $F = 0$, then, for the motion of the wheel at a velocity $c > c^*$, a moment

$$M = -\frac{r_1P^2}{2(k_1k_2)^{1/2}(u^4 - 1)^{1/2}} + o(\chi), \quad u = c/c^* > 1 \tag{2.10}$$

has to be applied.

The magnitude of the deflection of the rail at the point of contact

$$W(\xi_0) = -\frac{2\chi P(d_1k_2 + d_2k_1)u}{(4k_1k_2)^{5/4}\rho^{1/2}(u^4 - 1)^{3/2}} + o(\chi)$$

is negative. The deflection tends to zero as u^{-5} when $u \to \infty$ and is equal to zero when $\chi = 0$. When $\chi \to 0$, relations (2.8) are represented in the form

$$W(\xi)=\frac{P}{k_1\omega_1\omega_2(\omega_2^2-\omega_1^2)}\times\begin{cases}\omega_2\sin\omega_1(\xi-\xi_0), & \xi<\xi_0\\ \omega_1\sin\omega_2(\xi-\xi_0), & \xi\geqslant\xi_0\end{cases}$$

The moment (2.10) determines the wave drag and is necessary to maintain the steady motion of the wheel at a velocity greater than the critical velocity. It is not equal to zero when there are no dissipative forces. We note that, in the case of motion at velocities less than the critical velocity, no forces need be applied.

3. DISSIPATION OF ENERGY AND RESISTANCE TO MOTION

We shall consider the case of subcritical velocities ($c < c^*$) and write down a theorem on the change in the total mechanical energy in the steady-state case

$$d(T+\Pi)/dt=-2D[w^{\cdot}] \tag{3.1}$$

According to (1.1), the left-hand side of Eq. (3.1) is equal to $Mc/r_1 - Fc$ and the right-hand side is equal to $-2c^2D[W']$. As above, let the motion occur solely under the action of a moment ($F = 0$). Then, from (3.1), we obtain

$$M=-cr_1\chi\int_{-\infty}^{+\infty}(d_1W'''^2+d_2W'^2)d\xi \tag{3.2}$$

The shape of the deformed rail must be found from the solution of the boundary-value problem (2.2). However, by taking account of the smallness of the dissipation coefficient, χ, this problem may be replaced by problem (2.3), when the shape of the deformed rail is determined without taking account of the small dissipative forces. It is obvious that the addition of the forces introduces a distortion in the shape of the order of χ and, when the integral in (3.2) is calculated, the moment of the forces M will be found with an accuracy up to terms of the order of χ^2.

We will calculate the integral on the right-hand side of (3.2) using formulae (2.7). The calculation is conveniently carried out in complex form

$$\int_{-\infty}^{+\infty}(d_1W'''^2+d_2W'^2)d\xi=\frac{P^2}{8k_1^2\varepsilon^2\omega^2(\varepsilon^2+\omega^2)^2}\int_{-\infty}^{0}(d_1f'''^2+d_2f'^2)d\xi$$

$$f(\xi)=\tfrac{1}{2}\{(\omega+i\varepsilon)\exp[(\varepsilon+i\omega)\xi]+(\omega-i\varepsilon)\exp[(\varepsilon-i\omega)\xi]\} \tag{3.3}$$

By (2.4), we obtain

$$\varepsilon=[\tfrac{1}{2}\gamma(1-u^2)]^{1/2},\quad \omega=[\tfrac{1}{2}\gamma(1+u^2)]^{1/2},\quad \varepsilon^2+\omega^2=\gamma,\quad \gamma=(k_2/k_1)^{1/2}$$

The dimensionless velocity of the wheel u varies over a range [0, 1) in the subcritical case. Further, on evaluating the integral in (3.3), by (3.2) we obtain

$$M=-\frac{\chi r_1P^2(d_1k_2(3-2u^2)+d_2k_1)u}{8\rho^{1/2}k_1^{3/2}k_2(1-u^2)^{3/2}},\quad 0\leqslant u<1 \tag{3.4}$$

The moment of the resistance to the rolling of the wheel is equal, by definition, to the moment of the active forces applied to the wheel with the opposite sign, and the resistance force is equal to the moment of the resistance divided by r_1. The moment of the active forces is determined by formulae (2.1) and (3.4), which no longer hold in the resonance domain since, on approaching the critical velocity c^*, the quantity $W'(\xi_0) = \xi_0/r_1$ becomes large and the hypotheses incorporated into the model are violated.

It should be pointed out that the resistance to the rolling of the wheel in the low-velocity domain ($u < 1$) is proportional to the dissipative forces and disappears if these forces are equal to zero. When the wheel rolls at velocities exceeding the critical velocities, the resistance to rolling has a wavy form, and its dependence on the dissipative forces does not appear in the first approximation (there are no terms of the order of χ).

The dependence of the displacement of the wheel on the parameters of the steady-state motion

$$y_1-r_1=\begin{cases}-\dfrac{P\sqrt{2}}{4k_1^{1/4}k_2^{3/4}(1-u^2)^{1/2}}+O(\chi), & 0\leqslant u<1\\[2ex] -\dfrac{r_1P^2}{4k_1k_2(u^2-1)}+O(\chi), & u>1\end{cases} \tag{3.5}$$

is also of interest.

It is clear that formulae (3.5) cannot be used at velocities close to resonance ($u = 1$) and also in the case of large loads when the curvature of the rail at the point of contact is greater than r_1^{-1}, which contradicts the hypotheses incorporated in the model.

In concluding, we note that, as the wheel rolls along the rail, the points of the rail execute oscillatory motions with frequencies $c\omega$ in the subcritical case ($c < c^*$) and $c\omega_1, c\omega_2$ when the motion occurs at velocities greater than the critical velocities ($c > c^*$) since $w(s, t) = W(s - ct)$. This leads to vibrations of the air with the above-mentioned frequencies which should be treated as regular noise when a wheel rolls along a deformable rail.

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